Exercise 32

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x \to 2} x^3 = 8$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad 0 < |x - 2| < \delta \qquad \text{then} \qquad |x^3 - 8| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x - 2|$.

$$|x^3 - 8| < \varepsilon$$

$$|(x^2 + 2x + 4)(x - 2)| < \varepsilon$$

$$|x^2 + 2x + 4||x - 2| < \varepsilon$$

On an interval centered at $x = 2$, a positive constant C can be chosen so that $|x^2 + 2x + 4| < C$.

$$C|x - 2| < \varepsilon$$

$$|x - 2| < \frac{\varepsilon}{C}$$

To determine C, suppose that x is within a distance a from 2.

$$|x - 2| < a$$

$$-a < x - 2 < a$$

$$-a + 2 < x < a + 2$$

$$|x^2 + 2x + 4| < (a + 2)^2 + 2(a + 2) + 4 = a^2 + 6a + 12$$

The constant C is then $a^2 + 6a + 12$. Choose δ to be whichever is smaller between a and $\varepsilon/(a^2 + 6a + 12)$: $\delta = \min\{a, \varepsilon/(a^2 + 6a + 12)\}$. Now, assuming that $|x - 2| < \delta$,

$$\begin{aligned} |x^3 - 8| &= |(x^2 + 2x + 4)(x - 2)| \\ &= |x^2 + 2x + 4||x - 2| \\ &< (a^2 + 6a + 12)\left(\frac{\varepsilon}{a^2 + 6a + 12}\right) = \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to 2} x^3 = 8.$$